# Week 5: Expected value and Betting systems

**Random variable** A random variable represents a "measurement" in a random experiment. We usually denote random variable with capital letter $X, Y, \cdots$. If $S$ is the sample space of the experiment then to each $i \in S$ the random variable $X$ assigns a certain value $\alpha$ ( a real number). The random variable is described by its probability distribution

$$P(X = \alpha)$$

for all possible values $\alpha$ that the random variable can take. Of course we have

$$0 \le P(X = \alpha) \le 1 \quad \sum_{\alpha} P(X = \alpha) \, = \, 1$$

**Example:** If you roll a pair of dice consider the random variable

$$X = \text{ sum of the two dice}$$

Then $X$ takes values $2, 3, \cdots, 12$ and $P(X = 2) = 1/36$, $P(X = 3) \, = \, 2/36$ etc.. ■

**Expected value of a random variable** For a random variable $X$ the *expected value of* $X$ is the *average value* of $X$ which we denote by $E[X]$. It is given by

$$\boxed{\textbf{Expected value of } X : \quad E[X] \, = \, \sum_{\alpha} \alpha \, P(X = \alpha)}$$

**Chuck-a-luck** This game (found in fairgrounds) is played by rolling 3 dice and betting on a number between 1 and 6. You win your bet multiplied by the number of times your chosen appear on the the three dice. For example if you bet \$1 on 5 and roll $4, 5, 5$ you win \$2. A quick look at this game may make it appear reasonably fair. Since you roll 3 dice and there seems to be a probability $1/2$ that your chosen number appears and so the odds should be in your favor. For a second look let us compute your expected gain $E[W]$ at this game. Supose you bet on 5

$$\begin{aligned} E[W] &= 3P(3 \text{ fives}) + 2P(2 \text{ fives}) + 1P(1 \text{ five}) - 1P(0 \text{ five}) \\ &= 3 \cdot \frac{1}{216} + 2 \cdot \frac{15}{36} + 1 \cdot \frac{75}{36} - 1 \cdot \frac{125}{216} \, = \, -\frac{17}{216} \, = \, -.079 \end{aligned}$$

That is you loose around 8 cents on the dollar at this game. ∎

**Example: Expected gain at roulette**. At the Las Vegas roulette (with 38 numbers, 0,00,1,2,3, etc) you can do various bets (let's say the bet size is $1).

1. Bet on red (or black) and a successful bet pays you $1.
2. Bet on a number and a successful bet pays you $35.
3. Bet on the first (or the second, the third) dozen of numbers a successful bet pays you $2.
4. etc $\cdots$

You can find the list of all bets and payouts for Las Vegas and Monte-Carlo roulette at `http://en.wikipedia.org/wiki/Roulette`

You may wonder which of these bets is the more advantageous and so we compute the *expected gain* $E[W]$ for each bet

$$
\begin{aligned}
\text{Red} \quad E[W] &= 1 \cdot \frac{18}{38} - 1\frac{20}{38} = -\frac{2}{38} = -0.0526 \\
\text{Number} \quad E[W] &= 35 \cdot \frac{1}{38} - 1\frac{37}{38} = -\frac{2}{38} \\
\text{Dozen} \quad E[W] &= 2 \cdot \frac{12}{38} - 1\frac{26}{38} = -\frac{2}{38}
\end{aligned}
$$

All these bets (and all the other ones) are devised to give the same odds. It does not matter how you play, you shall loose on average around 5.3 cents for each dollar you bet.

If you bet on a group of $n$ numbers then the payout is

$$
\text{Payout for a bet on } n \text{ numbers} = \frac{36}{n} - 1
$$

and for such a bet the expected gain is

$$
E[W] = \left(\frac{36}{n} - 1\right) \cdot \frac{n}{38} - 1 \cdot \frac{38-n}{38} = \frac{36}{38} - 1 = -\frac{2}{38}
$$

∎

**Keno 10 spot card** Recall that in Keno the casino draws 20 numbers randomly out of 80 numbers. In a $m$ spot card you pick $m$ numbers and if $k$ of your $m$ numbers match the casino numbers you have a "catch of $k$". We have

$$
P(\text{catch of } k) = \frac{\binom{10}{k}\binom{70}{20-k}}{\binom{80}{20}}
$$

The probability and payouts for Keno vary a bit from place to place: for a 10 spot card the payouts by the Massachusetts lottery are (see `http://www.masslottery.com/games/keno.html` for all the payouts )

| Match | Payout |
|---|---|
| 10 | 100'000 |
| 9 | 10'000 |
| 8 | 500 |
| 7 | 80 |
| 6 | 20 |
| 5 | 2 |
| 0 | 2 |

So for a bet of 1 dollar the expected amount paid by the lottery is

$$\begin{aligned} E[W] &= 100000 \cdot \frac{\binom{10}{10}\binom{70}{10}}{\binom{80}{20}} + 10000 \cdot \frac{\binom{10}{9}\binom{70}{11}}{\binom{80}{20}} + 500 \cdot \frac{\binom{10}{8}\binom{70}{12}}{\binom{80}{20}} + 80 \cdot \frac{\binom{10}{7}\binom{70}{13}}{\binom{80}{20}} \\ & \quad +20 \cdot \frac{\binom{10}{6}\binom{70}{2014}}{\binom{80}{20}} + 2 \cdot \frac{\binom{10}{5}\binom{70}{15}}{\binom{80}{20}} + +2 \cdot \frac{\binom{10}{0}\binom{70}{20}}{\binom{80}{20}} \\ &= 0.0112211 \quad + \quad 0.0612064 \quad + \quad 0.0677096 \quad + \quad 0.1288914 \\ & \quad +0.2295878 \quad + \quad 0.1028553 \quad + \quad 0.0915814 \\ &= 0.6930534 \end{aligned} \tag{1}$$

The lottery keeps then more than 30 cents (!) of each dollar played on Keno and the number above tells you where the payout are. For example 22 cents on a dollar are given as payout for a catch of 6 while only about one cent as payout for a catch of 10, and so on.. ∎

**The martingale betting system**. Let us explain this betting system by an example. You just receive the news that you inherited from a long lost relative the nice sum of $\$2,550,000 = (2^8 - 1) \cdot 10,000)$. You move immediately to Atlantic city and devise the following gambling scheme. Every month you go to the craps table and bet $\$10,000$. If you win, you just won $\$10,000$ and you quit and live off your money for a month. Now if you loose play again you double your bet to $\$20,000$. If you win your second bet then your net win is $\$20,000 - \$10,000 = \$10,000$. Again if you loose you double you bet, etc.... In any case if you win the $k^th$ bet then your net gain is

$$10'000\left[-1 - 2 - 2^{k-1} + 2^k\right] = 10'000$$

using the geometric series $1 + x + \cdots + x^{k_1} = \frac{1-x^k}{1-x}$ with $x = 2$. So with this betting system, known as the *martingale* you do win $\$10'000$ every time.

If you have unlimited resources (and if the casino has no betting limit) you could in principle make money using subfair games. But of course none of this condition is true. If the probability to lose any single game is $q$ then

$$\text{Probability to lose everything } = q^8$$

since to lose everything you need to loose 8 times in a row. Let us compute the expected gain $W$ playing the game this way. We have

$$\boxed{E[W] \,=\, 10'000 \cdot (1 - q^8) - (2^8 - 1)10'000 \cdot p^8 \,=\, 10'000 \left[1 - (2q)^8\right]}$$

If the game were fair $p = 1/2$ then the probability to lose everything on a single month is $1/256 = 0.0039$ and the expected gain is 0. If you play craps for which $q = 251/295$ then the probability to lose everything in a single run 0.0044 and the expected gain is $-\$1188.92$.

The idea behind this betting (and many other) betting system is to make sure that you win (a little) with high probability

**Exercises:**

**Exercise 1:** The probability to win at powerball have been computed in a previous exercise (you can find them at `http://www.powerball.com/powerball/pb_prizes.asp`). Remember that a ticket is worse \$2. The expected amount paid back by the lottery for a bet depends on the jackpot. Today (March 1, 2013) the jackpot is \$103 million with a cash value of \$64.2 million. Compute your expected gain in the following three cases.

1. You take the cash value
2. You choose delayed payments
3. You take the cash value but you remember that actually you are going to pay roughly around 40% of federal and state taxes on your prize.

**Exercise 2:** Poker dice is a carnival game played with 5 dice with 9, 10, $J$, $Q$, $K$, $A$ on the sides instead of the usual 'numbers. The bettor chooses two different faces (let us say he chooses $Q$ and $K$) from the six choices and is paid at rate of 1:1 if both sides appear (that is they appear at least once). Compute your expected gain at this game. *Hint:* Compute the probability not to win and use the formula for $P(A \cup B)$.

**Exercise 3:** An *annuity* is the promise by $A$ (typically a life insurance company or a pension fund) to pay $B$ (typically an individual) a certain sum of money for the rest of his life. For example in many pensions systems, when you retire, you are promised a fixed amount of money every year for the rest of your life, depending on how much you saved throughout your life (and probably your age too).

To compute the value of an annuity usually one use

$$\boxed{\text{Value of the annuity} = \text{Expected amount paid to B by A}}$$

1. Assume that $A$ promises to pay $B$ \$10000 every year until $B$'s death. Based on public health data $A$ estimate that the probability that $B$ dies in any single year is $1 - p$. Assume that the first payment $X_0 = 1$ is made at once and we denote by $X_j$ the amount paid at the beginning of year $j$. Compute $E[X_j]$ and then the total value of the annuity.

2. In part 1. we have neglected the fact that money can earn interest. Suppose $A$ invests its money in some bonds which yields a fixed percentage of $\%\alpha$ per year. If $A$ needs to pay \$10000 unit to $B$ in year $i$, how much money does $A$ needs to have right now? Based on this compute the value of the annuity, that is how much money needs $A$ to have on hand at year 0 to pay the annuity in the future.

3. What is unrealistic in our model?

**Exercise 4:** Consider the martingale betting systems where the probability of winning a single game is $p$ and your limited fortune (or the house limit) allows to you bet at most $k$ times.

1. What is the probability that you loose everything exactly on your $n^th$ visit to the casino?

2. Compute the expected number of visits to the casino?
   *Hint:* How do you compute $1 + 2x + 3x^2 + 4x^3 + \cdots$?